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The states of W-class as shared resources for perfect teleportation and superdense coding

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As we know, the states of triqubit systems have two important classes: GHZ-class and W-class. In this paper, the states of W-class are considered for teleportation and superdense coding, and are generalized to multi-particle systems. First we describe two transformations of the shared resources for teleportation and superdense coding, which allow many new protocols from some known ones for that. As an application of these transformations, we obtain a sufficient and necessary condition for a state of W-class being suitable for perfect teleportation and superdense coding. As another application, we find that state $|W\rangle_{123} = \frac{1}{2}(|100\rangle_{123} + |010\rangle_{123} + \sqrt{2}|001\rangle_{123})$ can be used to transmit three classical bits by sending two qubits, which was considered to be impossible by P. Agrawal and A. Pati [Phys. Rev. A to be published]. We generalize the states of W-class to multi-qubit systems and multi-particle systems with higher dimension. We propose two protocols for teleportation and superdense coding by using W-states of multi-qubit systems that generalize the protocols by using $|W\rangle_{123}$ proposed by P. Agrawal and A. Pati. We obtain an optimal way to partition some W-states of multi-qubit systems into two subsystems, such that the entanglement between them achieves maximum value.

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I. INTRODUCTION

Dür *et al* [5] pointed out that the states of three qubits can be entangled in two inequivalent ways: GHZ-class and W-class. States in one class can not be converted from states in the other class by stochastic local operations and classical communication (SLOCC) [3]. With respect to loss of qubits, the two classes are rather different. The states of W-class are robust against loss of qubits (i.e., if we trace out any one qubit, then there is some genuine entanglement between the remaining two qubits), while GHZ-states are not. The GHZ-class has been extensively studied from many aspects, while W-class may need more effort to clearly characterize it.

Recently, more attentions have been taken on the states of W-class. They have been considered for many important quantum information processing tasks [10, 11, 12, 13, 14]. W-states were considered as quantum channel for teleportation of entangled pairs in [10]. Probabilistic teleportation of a qubit state via a W-state was studied in [11]. Furthermore, in [12] the authors discovered a subclass of W-class suitable for perfect teleportation and superdense coding, but they did not give the sufficient and necessary condition for a W-state being suitable for that. In addition, W-class has been used for quantum key distribution [13], and in illustrating violation of local realism [14]. At the same time, there have been various proposals to prepare W-states [21]. Considering the importance of W-class, in this paper we focus on the states of W-class for perfect teleportation and superdense coding.

As we know, quantum teleportation and superdense coding are two amazing and interesting processes in quantum information theory, where entangled states as shared resources play a crucial role [1, 2]. The two processes have a close relationship that has been investigated by Werner [4]. In the original protocols [1, 2] for teleportation and superdense coding, EPR pairs were considered as the shared resources. Latter, more candidates have been considered. For instance, maximally entangled states of triparticle were considered in [7, 8, 9], and maximally entangled states of multi-particle were considered in [17, 18, 19]. Specially, non-maximally entangled states have been considered for teleportation and superdense coding by many authors. For instance, non-maximally entangled states have been considered for probabilistic teleportation [22, 23, 24, 25] and probabilistic superdense coding [29, 30]. In addition, the classical information capacity of deterministic superdense coding with non-maximally entangled states has been studied by [26, 27, 28]. Recently, various kinds of quantum channels have been explored for deterministic and unambiguous superdense coding [31, 32, 33].

Observing the states of triparticle, interestingly we find that some states from GHZ-class are suitable for perfect teleportation and superdense coding, but some states are not. For instance, there are two states of GHZ-class:

$$|GHZ\rangle_{123} = \frac{1}{\sqrt{2}}(|000\rangle_{123} + |111\rangle_{123}), \qquad (1)$$

$$|\widetilde{GHZ}\rangle_{123} = \frac{1}{\sqrt{3}}(\sqrt{2}|000\rangle_{123} + |111\rangle_{123}). \qquad (2)$$

It is well known that the first state can be used for perfect teleportation and superdense coding. But the second one can not. At the same time, similar case is found in W-

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class. For the following two states of W-class:

$$|W\rangle_{123} = \frac{1}{2}(|100\rangle_{123} + |010\rangle_{123} + \sqrt{2}|001\rangle_{123}), \quad (3)$$

$$|\widetilde{W}\rangle_{123} = \frac{1}{\sqrt{3}}(|100\rangle_{123} + |010\rangle_{123} + |001\rangle_{123}), \quad (4)$$

the first state can be used for teleportation of a qubit state and for superdense coding of two classical bits by one qubit as shown in [12], but the second state can not.

In this paper, we focus on the states of W-class. Firstly we have a brief analysis on the above two states from W-class, to find out what difference lies between them. We can partition the three particles whose state is $|\widetilde{W}\rangle_{123}$ into two subsystems in three ways: 1|23, 2|13, and 12|3. Then we have

$$\begin{aligned} \rho_1 = \rho_2 = \rho_3 &= \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|, \\ \rho_{23} = \rho_{13} = \rho_{12} &= \frac{2}{3}|\psi\rangle\langle\psi| + \frac{1}{3}|00\rangle\langle 00|, \end{aligned} \quad (5)$$

where $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ is an EPR pair. If we calculate the entanglement between the two subsystems resulted from the above partitioning ways, we have

$$E_{1(23)}(|\widetilde{W}\rangle_{123}) = E_{2(13)}(|\widetilde{W}\rangle_{123}) = E_{(12)3}(|\widetilde{W}\rangle_{123}) < 1. \quad (6)$$

Note that here we adopt the definition of partial entropy entanglement, where the entanglement between subsystems A and B involved in the pure state $|\Psi\rangle_{AB}$ is defined as

$$E_{A|B}(|\Psi\rangle_{AB}) = S(\rho_A) \quad (7)$$

where $\rho_A = tr_B(|\Psi\rangle\langle\Psi|)$ and $S(\rho_A)$ is the von Neumann entropy [34].

Now the above results show that: (i) State $|\widetilde{W}\rangle_{123}$ has symmetry property that no matter how we partition it into two subsystems, the results are always the same. (ii) The entanglement between any two subsystems resulted from the above partition is less than one ebit, and therefore it is not suitable for perfect teleportation and superdense coding. (iii) If any qubit of the state is lost, the residual two qubits still share genuine entanglement between them, which is called the *robustness* against loss of qubits. Contrary to that, the states of GHZ-class do not have this robustness.

For state $|W\rangle_{123}$, we can also partition it into two subsystems by the same way stated before, and then we have

$$\rho_3 = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|, \ \rho_{12} = \frac{1}{2}|\psi\rangle\langle\psi| + \frac{1}{2}|00\rangle\langle 00|, \quad (8)$$

and

$$\begin{aligned} \rho_1 = \rho_2 &= \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|, \\ \rho_{23} = \rho_{13} &= \frac{3}{4}|\phi\rangle\langle\phi| + \frac{1}{4}|00\rangle\langle 00|, \end{aligned} \quad (9)$$

where $|\phi\rangle = \frac{|10\rangle + \sqrt{2}|01\rangle}{\sqrt{3}}$ is a partial entangled pair. Calculating the entanglement between any two subsystems resulted, we have

$$E_{(12)3}(|W\rangle_{123}) = 1, E_{1(23)}(|W\rangle_{123}) = E_{2(13)}(|W\rangle_{123}) < 1. \quad (10)$$

From the above results we know that: (i) If we partition the state in the way: 12|3, then the two subsystems resulted share an entangled state with one ebit of entanglement. Therefore, if Alice has particle '3' and Bob has particles '1' and '2', then Alice can send Bob two classical bits of information by sending one qubit, or Bob can teleport a state of qubit to Alice as shown in [12]. (ii) The state does not have symmetry property, and thus the other two partitioning ways lead to two subsystems whose entanglement is less than one ebit. (iii) The state also has the robustness against loss of qubits.

In summary, from the above considerations, we have seen that except for the robustness against loss of qubits, the states of W-class have also some other interesting properties. For instance, state $|\widetilde{W}\rangle_{123}$ has symmetry property that may be very useful for some applications, but the entanglement involved does not attain one ebit, which, on the other hand, may be its limitation. Oppositely, state $|W\rangle_{123}$, without symmetry property, can be used as a maximally entangled state if we choose an appropriate way to partition it. Therefore, the states of W-class like $|W\rangle_{123}$ and $|\widetilde{W}\rangle_{123}$ may be worthy of further consideration, and we would like to do that in this paper.

In this paper, based on the previous works [5, 12], we characterize further the states of W-class, and consider further the protocols for perfect teleportation and superdense coding via W-class. In Section II, our main aim is to find what states of W-class can be used for teleportation and superdense coding. Firstly, in Sec. II A and II B, we describe two transformations of the shared resources for teleportation and superdense coding, which will allow many new protocols for that from some known protocols. In Sec. II C, as an application, we obtain a sufficient and necessary condition for a state of W-class being suitable for perfect teleportation and superdense coding. In Sec. II D, as another application, we find that state $|W\rangle_{123}$ can be used to transmit three classical bits by sending two qubits. In Section III, our aim is to generalize states $|W\rangle_{123}$ and $|\widetilde{W}\rangle_{123}$ to multi-particle systems, and characterize them. In Sec. III A, we generalize states $|W\rangle_{123}$ and $|\widetilde{W}\rangle_{123}$ to multi-qubit systems. In Sec. III B, we propose two protocols for teleportation and superdense coding that generalize the protocols by using state $|W\rangle_{123}$ indicated in [12]. In Sec. III C, for the states that generalize $|\widetilde{W}\rangle_{123}$ to N-qubit systems, we point out an optimal way to partition them into two subsystems such that the two subsystems resulted share maximum entanglement, and, when N is an even number, in a certain sense these generalized states can be exploited as maximally entangled states. In Sec. III D, we generalize state $|W\rangle_{123}$ to multi-particle systems with higher dimension.

Finally, some concluding remarks are made in Sec. IV.

II. PROTOCOLS FOR SUPERDENSE CODING AND TELEPORTATION

In this section, firstly we will describe two transformations of the shared resources for superdense coding and teleportation. Then as an application of these transformations, we will obtain a sufficient and necessary condition for a state of W-class being suitable for perfect teleportation and superdense coding. Also, we will find that state $|W\rangle_{123}$ can be used to transmit three classical bits by sending two qubits, which was considered to be impossible in [12].

A. Transformations of the shared resources for superdense coding

The standard protocol for superdense coding can be described in the following. Suppose that Alice possesses subsystem A and Bob possesses subsystem B and the two subsystems share an entangled state $|\varphi\rangle_{AB}$. Alice can apply operators from the set of unitary operators $\{U_A^x\}$ on her subsystem A, and then send A to Bob. Then the states of bisystem AB belonging to Bob will form an orthogonal set $\{|\Phi_x\rangle\}$. That can be described as

$$(U_A^x \otimes I_B)|\varphi\rangle_{AB} = |\Phi_x\rangle. \tag{11}$$

Because of the orthogonality of set $\{|\Phi_x\rangle\}$, Bob can make a projective measurement on system AB with projectors $P_x = |\Phi_x\rangle\langle\Phi_x|$ to perfectly distinguish the set $\{|\Phi_x\rangle\}$, such that Bob can know exactly which operator Alice has applied. In this process, if the number of the operators in set $\{|\Phi_x\rangle\}$ is N, then Bob can get $\log_2 N$ classical bits of information from Alice.

If we take a viewpoint from discrimination between unitary operations [15, 16], then the core of superdense coding is to find out as many unitary operators as possible, such that they can be perfectly discriminated by state $|\varphi\rangle_{AB}$.

Now suppose there is another state $|\varphi^{'}\rangle_{AB}$ given by

$$|\varphi^{'}\rangle_{AB} = (U_A \otimes I_B)|\varphi\rangle_{AB} \tag{12}$$

where U_A is a unitary operator acting on subsystem A and I_B is identity operator on B. Then we have

$$\begin{aligned}(U_A^x U_A^\dagger \otimes I_B)|\varphi^{'}\rangle_{AB} &= (U_A^x U_A^\dagger \otimes I_B)(U_A \otimes I_B)|\varphi\rangle_{AB} \\ &= (U_A^x \otimes I_B)|\varphi\rangle_{AB} \\ &= |\Phi_x\rangle.\end{aligned} \tag{13}$$

Thus it is shown that by sharing state $|\varphi^{'}\rangle_{AB}$, Alice and Bob can also fulfill the same task as by sharing state $|\varphi\rangle_{AB}$, and the only thing changed is that the set of operators applied by Alice turns to $\{U_A^x U_A^\dagger\}$.

We can also take a unitary operation V_B on subsystem B getting a new entangled state in the following way:

$$|\varphi^{''}\rangle_{AB} = (I_A \otimes V_B)|\varphi\rangle_{AB}. \tag{14}$$

Then we have

$$\begin{aligned}(U_A^x \otimes I_B)|\varphi^{''}\rangle_{AB} &= (U_A^x \otimes I_B)(I_A \otimes V_B)|\varphi\rangle_{AB} \\ &= (I_A \otimes V_B)(U_A^x \otimes I_B)|\varphi\rangle_{AB} \\ &= (I_A \otimes V_B)|\Phi_x\rangle.\end{aligned} \tag{15}$$

From the orthogonality of set $\{|\Phi_x\rangle\}$ and the unitarity of V_B, the states in set $\{(I_A \otimes V_B)|\Phi_x\rangle\}$ are clearly mutually orthogonal. Therefore, by sharing state $|\varphi^{''}\rangle_{AB}$, Alice and Bob can also fulfill the same task as by sharing $|\varphi\rangle_{AB}$. The only change is that Bob should make a projective measurement on AB with projectors $P_x = (I_A \otimes V_B)|\Phi_x\rangle\langle\Phi_x|(I_A \otimes V_B^\dagger)$.

It is worth pointing out that transformations (12) and (14) can occur simultaneously. Then we should combine (13) and (15) getting a new protocol. For simplicity, we describe them separately. Similar case will occur in the protocol for teleportation as we will show in Sec. II B.

B. Transformations of the shared resources for teleportation

The standard protocol for teleportation can be described as follows. At first, Alice and Bob share an entangled state $|\varphi\rangle_{AB}$ of bisystem AB, of which subsystem A belongs to Alice and subsystem B belongs to Bob. Also Alice possesses another system 'a' whose state is $|\Psi\rangle$. Now Alice's task is to send the state $|\Psi\rangle$ to Bob such that the state of Bob's subsystem turns to that, with the help of the shared state $|\varphi\rangle_{AB}$ and some classical communication between them. This task can be fulfilled, if $|\Psi\rangle_a|\varphi\rangle_{AB}$ can be rewritten in the form

$$|\Psi\rangle_a|\varphi\rangle_{AB} = \frac{1}{\sqrt{D}}\sum_{x=1}^{D}|\Phi_x\rangle_{aA}U_x|\Psi\rangle_B \tag{16}$$

where $\{|\Phi_x\rangle_{aA}\}$ is a set of mutually orthogonal states of joint system 'aA', and $\{U_x\}$ is a set of unitary operators on subsystem B. Alice can now fulfill her task by two steps:

- Make a projective measurement on 'aA' in a basis that includes $\{|\Phi_x\rangle_{aA}\}$, getting measurement result 'x' with probability $\frac{1}{D}$.
- Send the result 'x' to Bob who applies a unitary operator $U_x^\dagger$ on B, and thus recovers the state $|\Psi\rangle$ on subsystem B.

Now we assume that the shared resource is $|\varphi^{'}\rangle_{AB}$ given by Eq. (12). Then we have

$$\begin{aligned}|\Psi\rangle_a|\varphi^{'}\rangle_{AB} &= (I_a \otimes U_A \otimes I_B)|\Psi\rangle_a|\varphi\rangle_{AB} \\ &= \frac{1}{\sqrt{D}}\sum_{x=1}^{D}(I_a \otimes U_A)|\Phi_x\rangle_{aA}U_x|\Psi\rangle_B.\end{aligned} \tag{17}$$

The above process shows that if Alice can fulfill her task by sharing state $|\varphi\rangle_{AB}$, then she can also do that by sharing state $|\varphi^{'}\rangle_{AB}$, with a projective measurement on her system 'aA' in a basis that includes $\{(I_a\otimes U_A)|\Phi_x\rangle_{aA}\}$.

Similarly, Alice and Bob can also share the state $|\varphi^{''}\rangle_{AB}$ given by Eq. (14). Then we have

$$
\begin{aligned}
|\Psi\rangle_a|\varphi^{''}\rangle_{AB} &= (I_a\otimes I_A\otimes V_B)|\Psi\rangle_a|\varphi\rangle_{AB}\\
&= \frac{1}{\sqrt{D}}\sum_{x=1}^{D}|\Phi_x\rangle_{aA}(V_BU_x)|\Psi\rangle_B.
\end{aligned}
\tag{18}
$$

This shows that by sharing state $|\varphi^{''}\rangle_{AB}$, Alice and Bob can fulfill the same task as by sharing state $|\varphi\rangle_{AB}$, with the only change that Bob's operator set turns to $\{U_x^\dagger V_B^\dagger\}$.

C. Application I: perfect teleportation of qubit states and superdense coding of two classical bits via states of W-class

Now there are two tasks described below. (i) Can Alice teleport a state of qubit to Bob by sharing a W-state between them? (ii) Can Alice send Bob two classical bits of information by sending only one qubit, with the help of a shared W-state? In Ref. [11], the authors showed that by sharing state $|\widetilde{W}\rangle_{123}$ between Alice and Bob, task (i) can be fulfilled in a probabilistic manner, but not in a perfect fashion. Also it is readily seen that the state can not be used for task (ii) in a perfect fashion. These results seem to imply that W-states are not suitable for perfect teleportation and superdense coding. However, Ref. [12] found that a class of states within W-class can be used to do that. Those states are given in the form

$$
\begin{aligned}
|W_n\rangle_{123} = \frac{1}{\sqrt{2+2n}}(&|100\rangle_{123}+\sqrt{n}e^{i\gamma}|010\rangle_{123}\\
&+\sqrt{n+1}e^{i\delta}|001\rangle_{123}).
\end{aligned}
\tag{19}
$$

A prototype state in this class is $|W\rangle_{123}$ given by Eq. (3). In the conclusions of [12], the authors proposed some problems worthy of further consideration, one of which is whether there are other classes of W-states useful for perfect teleportation and superdense coding. Now with those transformations we stated previously, indeed we can discover a class of W-states useful for perfect teleportation and superdense coding which includes these states given by Eq. (19). As a result, it will explain why state $|W_n\rangle_{123}$ can be used for tasks (i) and (ii).

As we know state $|GHZ\rangle_{123}$ is suitable for tasks (i) and (ii). Here we have a brief review of these protocols for that. Alice and Bob share state $|GHZ\rangle_{123}$, of which Bob has particle '3', and Alice has particles '1' and '2'. Also Alice has particle 'a' with state $|\Psi\rangle=\alpha|0\rangle+\beta|1\rangle$ that will be teleported from Alice to Bob. Then there is

$$
\begin{aligned}
&|\Psi\rangle_a|GHZ\rangle_{123}\\
&=\frac{1}{2}[|\psi_1^+\rangle_{a12}\otimes|\Psi\rangle_3+|\psi_1^-\rangle_{a12}\otimes\sigma_3|\Psi\rangle_3\\
&\quad+|\psi_2^+\rangle_{a12}\otimes\sigma_1|\Psi\rangle_3+|\psi_2^-\rangle_{a12}\otimes(i\sigma_2)|\Psi\rangle_3],
\end{aligned}
\tag{20}
$$

where $\{|\psi_1^\pm\rangle,|\psi_2^\pm\rangle\}$ are given by Eq. (25) in Sec. II D. Alice now makes a projective measurement on particles 'a12' in a basis including $\{|\psi_1^\pm\rangle,|\psi_2^\pm\rangle\}$, and then sends the measurement results to Bob who can recover state $|\Psi\rangle$ at particle '3' by applying appropriate operations. Therefore task (i) is fulfilled. For task (ii), similarly, $|GHZ\rangle_{123}$ is shared between Alice and Bob, and we let Alice have particle '3' and let Bob have the left particles. Then Alice first applies $\{I,\sigma_1,-i\sigma_2,\sigma_3\}$ on her qubit and then sends her qubit to Bob. As a result, the possible states of the three qubits holden by Bob will form an orthogonal set with four states that can be perfectly distinguished by Bob. Therefore Bob can get two classical bits from Alice.

Now with the above protocols and the transformations stated in Sec. II A and II B, we can obtain a subclass of W-class useful for perfect teleportation and superdense coding. We give our result as follows.

Theorem 1. A state of W-class is suitable for perfect teleportation and superdense coding if, and only if it can be converted from state $|GHZ\rangle_{123}$ by such a unitary operation that is the tensor product of a two-qubit unitary operation and a one-qubit unitary operation.

Proof. The sufficiency follows from the protocols by using $|GHZ\rangle_{123}$ and the transformations stated in Sec. II A and II B. We provide Fig. 1 and Fig. 2 to show that.

Next we verify the necessity. From our knowledge about teleportation and superdense coding, it follows that a state $|\varphi\rangle_{123}$ of triqubit that can be used for tasks (i) and (ii) can necessarily be partitioned into two subsystems, say A and B, which share one ebit of entanglement. That is, $|\varphi\rangle_{AB}$ can be regarded as a maximally entangled state of bisystem AB. In addition, we know that all the maximally entangled states of bisystem AB can be converted from each other by local unitary operations (i.e., in the form $U_A\otimes U_B$). Therefore, the necessity of the theorem follows. This completes the proof.

Now we have a brief test that the states given by Eq. (19) used in [12] for perfect teleportation and superdense coding are contained in the subclass stated by us. Firstly we can rewrite Eq. (19) in the following

$$
|W_n\rangle_{123}=\frac{1}{\sqrt{2}}(|\phi\rangle_{12}|0\rangle_3+e^{i\delta}|00\rangle_{12}|1\rangle_3),
\tag{21}
$$

where $|\phi\rangle=\frac{1}{\sqrt{1+n}}(|10\rangle+\sqrt{n}e^{i\gamma}|01\rangle)$. Then $|W_n\rangle_{123}$ can be converted from $|GHZ\rangle_{123}$ by

$$
|W_n\rangle_{123}=(V_{12}\otimes I_3)|GHZ\rangle_{123}
\tag{22}
$$

where V_{12} is a unitary operator acting on particles '1' and '2' given as

$$
V_{12}=|\phi\rangle\langle 00|+|11\rangle\langle 01|+|\phi^\perp\rangle\langle 10|+e^{i\delta}|00\rangle\langle 11|,
\tag{23}
$$

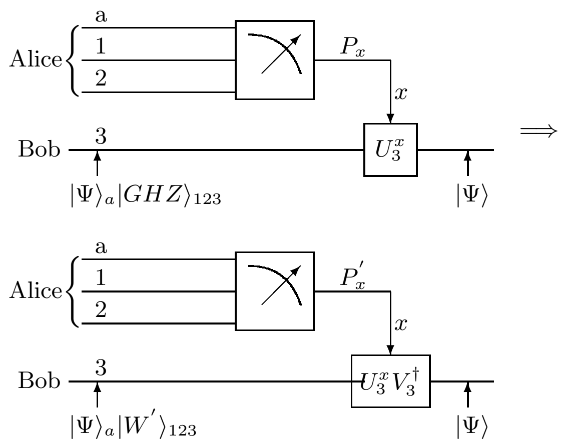

FIG. 1: Teleportation of state $|\Psi\rangle$: when the shared resource is changed from $|GHZ\rangle_{123}$ to $|W^{'}\rangle_{123} = (V_{12} \otimes V_3)|GHZ\rangle_{123}$, then the measurement projectors made by Alice are changed from $\{P_x\}$ to $\{P_x^{'} : (I_a \otimes V_{12})P_x(I_a \otimes V_{12}^\dagger)\}$, and the operators applied by Bob are changed from $\{U_3^x\}$ to $\{U_3^x V_3^\dagger\}$.

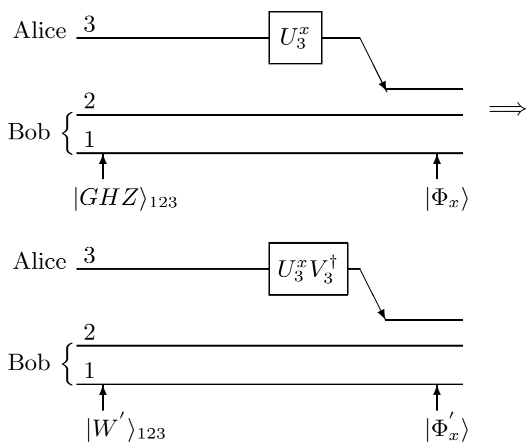

FIG. 2: Superdense coding of two classical bits: when the shared resource is changed from $|GHZ\rangle_{123}$ to $|W^{'}\rangle_{123} = (V_{12} \otimes V_3)|GHZ\rangle_{123}$, then the operators applied by Alice are changed from $\{U_3^x\}$ to $\{U_3^x V_3^\dagger\}$, and the states resulted at Bob' end are changed from $\{|\Phi_x\rangle\}$ to $\{|\Phi_x^{'}\rangle : (V_{12} \otimes I_3)|\Phi_x\rangle\}$.

where$|\phi^\perp\rangle = \frac{1}{\sqrt{1+n}}(\sqrt{n}e^{-i\gamma}|10\rangle - |01\rangle)$. Now we have shown that $|W_n\rangle_{123}$ satisfies the condition given in Theorem 1. Thus it is natural to use it for teleportation and superdense coding as did in [12]. In addition, as a special state in Eq. (19), $|W\rangle_{123}$ is of course suitable for that.

D. Application II: use state $|W\rangle_{123}$ to transmit three classical bits by sending two qubits

In Ref. [12] the authors thought that it may be not possible to use state $|W\rangle_{123}$ to transmit three classical bits by sending two qubits. Then they conjectured that there may exist some other states of W-class that can be used for such a task. In fact, state $|W\rangle_{123}$ can be used to do that as we will show below. Furthermore, there are indeed many W-states can be used for that.

Firstly we know that state $|GHZ\rangle_{123}$ can be shared by Alice and Bob, such that Alice can send Bob three classical bits of information by sending two qubits [7, 12]. In this case, Alice possesses the first two qubits, and the unitary operators applied by Alice can be in the set

$$\begin{aligned}\mathcal{U}_{12} = \{&I \otimes I, \sigma_1 \otimes I, (-i\sigma_2) \otimes I, \sigma_3 \otimes I,\\ &I \otimes \sigma_1, I \otimes (-i\sigma_2), \sigma_1 \otimes \sigma_1, \sigma_1 \otimes (-i\sigma_2)\}.\end{aligned} \tag{24}$$

After Alice's operation on her qubits and sending that to Bob, the state of the three qubits will be in one of the following states

$$\begin{aligned}|\psi_1^\pm\rangle &= \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle), \quad |\psi_2^\pm\rangle = \frac{1}{\sqrt{2}}(|100\rangle \pm |011\rangle),\\ |\psi_3^\pm\rangle &= \frac{1}{\sqrt{2}}(|010\rangle \pm |101\rangle), \quad |\psi_4^\pm\rangle = \frac{1}{\sqrt{2}}(|110\rangle \pm |001\rangle).\end{aligned} \tag{25}$$

Since the above eight states are mutually orthogonal, Bob can make a projective measurement to perfectly distinguish them, such that he can get three classical bits of information from Alice.

Now from the feasible protocol above, we can obtain that $|W\rangle_{123}$ is also suitable for transmitting three classical bits by sending two qubits. First of all, one should notice that

$$(U_{12} \otimes I_3)|GHZ\rangle_{123} = |W\rangle_{123} \tag{26}$$

where

$$U_{12} = |\varphi^+\rangle\langle 00| + |11\rangle\langle 01| + |\varphi^-\rangle\langle 10| + |00\rangle\langle 11|, \tag{27}$$

and $|\varphi^\pm\rangle = \frac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle)$. U_{12} is a joint unitary operator acting on the first two qubits, and can also be given by matrix form in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{28}$$

Now we let Alice and Bob share state $|W\rangle_{123}$, of which the first two qubits belong to Alice and the last qubit belongs to Bob. From Eqs. (12), (13) and (26), we soon get that if Alice chooses operators $U_x \in \mathcal{U}_{12}U_{12}^\dagger$ on her two qubits and then sends that to Bob, the eight orthogonal states given by Eq. (25) will appear at Bob's end. Therefore, Bob can make a projective measurement to perfectly distinguish which operator has been applied by Alice, and then gets three classical bits of information from Alice. The process is shown in Fig. 3.

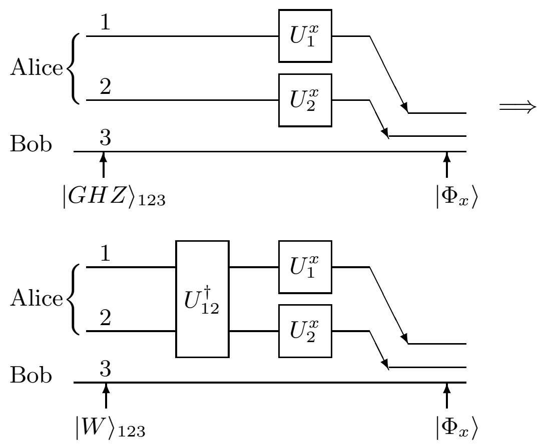

FIG. 3: Change the channel from $|GHZ\rangle_{123}$ to $|W\rangle_{123}$ for superdense coding of three classical bits.

Note that every operator in the set $\mathcal{U}_{12}$ can be chosen as tensor product of two local operators. Therefore, state $|GHZ\rangle_{123}$ can be suitable for the case of two senders and one receiver, where two senders, spatially separated, make only local operations on their own qubits. This case was generalized to the case of multi-sender vs one receiver in [17, 18, 19]. With respect to state $|W\rangle_{123}$, although it can be used to send three classical bits, it may be not suitable for the case of two senders and one receiver, because of the non-locality of U_x that derives from the non-locality of U_{12}. Indeed, it was shown in [5] that the states in W-class can not be converted from the states in GHZ-class by local operations. It is also worth pointing out that, here we just propose a scheme for sending three classical bits where non-local operators are allowed by the sender, but we do not prove the impossibility of local operators.

Note that all the states of W-class that satisfy the condition given by Theorem 1 in Sec. II C are suitable for the task stated in this subsection.

III. CHARACTERIZE W-CLASS OF MULTI-PARTICLE SYSTEMS

A. Generalize states $|W\rangle_{123}$ and $|\widetilde{W}\rangle_{123}$ to multi-qubit systems

Considering the special properties of states $|W\rangle_{123}$ and $|\widetilde{W}\rangle_{123}$, it is interesting to generalize them to multi-qubit systems. In fact, it is not difficult to do that. As shown in [5], state $|\widetilde{W}\rangle_{123}$ can be easily generalized to multi-qubit systems in the form ($N \geq 2$):

$$|\widetilde{W}^N\rangle = \frac{1}{\sqrt{N}}(|\overbrace{10\ldots0}^{N}\rangle + |01\ldots0\rangle + \cdots + |0\ldots01\rangle). \tag{29}$$

Clearly the state is symmetric in the sense that permutation of particles does not change state. Therefore, if we partition it into two subsystems in such a way: $(1\ldots i-1, i+1\ldots N)|i$ (simply denoted by $(\neq i)|i$), then we get the Schmidt decomposition form:

$$|\widetilde{W}^N\rangle = \sqrt{\frac{N-1}{N}}|\widetilde{W}^{N-1}\rangle|0\rangle_i + \sqrt{\frac{1}{N}}|0\ldots00\rangle|1\rangle_i. \tag{30}$$

Then two density operators associated with the two subsystems are

$$\begin{aligned} \rho_i &= \frac{1}{N}|1\rangle\langle1| + \frac{N-1}{N}|0\rangle\langle0|, \\ \rho_{(\neq i)} &= \frac{1}{N}|0\ldots0\rangle\langle0\ldots0| + \frac{N-1}{N}|\widetilde{W}^{N-1}\rangle\langle\widetilde{W}^{N-1}|, \end{aligned} \tag{31}$$

where $i = 1, 2, \ldots, N$, and $\rho_{(\neq i)}$ denotes the density operator of $|\widetilde{W}^N\rangle$ losing the ith qubit. Calculating the entanglement between the two subsystems, we have

$$E_{(\neq i)|i}(|\widetilde{W}^N\rangle) = -p\log_2 p - (1-p)\log_2(1-p) \leq 1, \tag{32}$$

where we let $p = \frac{1}{N}$. Now from the properties of binary entropy [34], we know that the value decreases while N increases, and the equality can be attained when $N = 2$. In fact, when $N = 2$, the state reduces to the EPR pair $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

State $|W\rangle_{123}$ can be generalized to multi-qubit systems in the following form:

$$\begin{aligned} |W^N\rangle = &\frac{1}{\sqrt{2(N-1)}}\Big(|\overbrace{10\ldots0}^{N}\rangle + \cdots + \\ &|0\ldots10\rangle + \sqrt{N-1}|0\ldots01\rangle\Big). \end{aligned} \tag{33}$$

This state is not fully symmetric. More exactly, permutation of the first $N-1$ qubits does not change state. If we partition the state into two subsystems in the way: $(1\ldots i-1, i+1\ldots N)|i$ as before, then when $i \neq N$ it can be rewritten in the Schmidt decomposition form:

$$\begin{aligned} |W^N\rangle = &\sqrt{\frac{2N-3}{2(N-1)}}|\varphi^{N-1}\rangle|0\rangle_i \\ &+ \frac{1}{\sqrt{2(N-1)}}|0\ldots00\rangle|1\rangle_i, \end{aligned} \tag{34}$$

where

$$\begin{aligned} |\varphi^{N-1}\rangle = &\frac{1}{\sqrt{2N-3}}\Big(|\overbrace{10\ldots0}^{N-1}\rangle + \cdots + |0\ldots10\rangle \\ &+ \sqrt{N-1}|0\ldots01\rangle\Big). \end{aligned} \tag{35}$$

Then the entanglement between the two subsystems resulted is always less than one ebit, and when $N = 2$ it attains one ebit.

When $i = N$, the state can be rewritten in the form of maximally entangled state

$$|W^N\rangle = \frac{1}{\sqrt{2}}|\widetilde{W}^{N-1}\rangle|0\rangle_N + \frac{1}{\sqrt{2}}|0\ldots 00\rangle|1\rangle_N. \qquad (36)$$

Therefore, if we partition $|W^N\rangle$ into two subsystems in the way: $(1\ldots N-1)|N$, then the entanglement between the two subsystems is: $E_{(1\ldots N-1)|N}(|W^N\rangle) = 1$. Thus the state can be used as a shared resource for teleportation and superdense coding, which will be detailed in the following subsection.

B. State $|W^N\rangle$ as a shared resource for teleportation and superdense coding

Here we see how state $|W^N\rangle$ is used for teleportation and superdense coding.

a. Teleportation Let Alice and Bob share state $|W^N\rangle$. Now Alice possesses the first $N-1$ qubits while Bob possesses the last qubit. Alice also has a particle 'a' in the unknown state $|\Psi\rangle_a = \alpha|0\rangle_a + \beta|1\rangle_a$. Then the combined input state can be rewritten as

$$\begin{aligned}
|\Psi\rangle_a|W^N\rangle &= (\alpha|0\rangle_a + \beta|1\rangle_a)\frac{1}{\sqrt{2}}(|\widetilde{W}^{N-1}\rangle|0\rangle + |0\ldots 00\rangle|1\rangle) \\
&= \frac{1}{\sqrt{2}}\Big[\alpha|0\rangle_a|\widetilde{W}^{N-1}\rangle|0\rangle + \alpha|0\rangle_a|0\ldots 00\rangle|1\rangle + \beta|1\rangle_a|\widetilde{W}^{N-1}\rangle|0\rangle + \beta|1\rangle_a|0\ldots 00\rangle|1\rangle\Big] \\
&= \frac{1}{2}\Big[(|\eta^+\rangle + |\eta^-\rangle)\alpha|0\rangle + (|\xi^+\rangle - |\xi^-\rangle)\alpha|1\rangle + (|\xi^+\rangle + |\xi^-\rangle)\beta|0\rangle + (|\eta^+\rangle - |\eta^-\rangle)\beta|1\rangle\Big] \\
&= \frac{1}{2}\Big[|\eta^+\rangle(\alpha|0\rangle + \beta|1\rangle) + |\eta^-\rangle(\alpha|0\rangle - \beta|1\rangle) + |\xi^+\rangle(\alpha|1\rangle + \beta|0\rangle) + |\xi^-\rangle(\beta|0\rangle - \alpha|1\rangle)\Big],
\end{aligned} \qquad (37)$$

where $\{|\eta^\pm\rangle, |\xi^\pm\rangle\}$ is a set of orthogonal states given by

$$\begin{aligned}
|\eta^\pm\rangle &= \frac{1}{\sqrt{2}}\Big(|0\rangle_a|\widetilde{W}^{N-1}\rangle \pm |1\rangle_a|0\ldots 00\rangle\Big), \\
|\xi^\pm\rangle &= \frac{1}{\sqrt{2}}\Big(|1\rangle_a|\widetilde{W}^{N-1}\rangle \pm |0\rangle_a|0\ldots 00\rangle\Big).
\end{aligned} \qquad (38)$$

Now Alice makes a projective measurement in a basis that includes the states $\{|\eta^\pm\rangle, |\xi^\pm\rangle\}$ on the particles possessed by herself. Then she sends the results of her measurement using two classical bits to Bob who can apply one of the unitary operators $\{I, \sigma_3, \sigma_1, -i\sigma_2\}$ to convert the state of his particle to that of particle 'a'. Now the teleportation protocol has been completed. This protocol consumes one ebit of shared entanglement and two bits of classical communication between Alice and Bob.

b. Superdense coding Let Alice and Bob share state $|W^N\rangle$. If Bob possesses the first $N-1$ qubits and Alice possesses the last qubit, then the coding protocol is described below:

$$\begin{aligned}
00:\ |W^N\rangle &\xrightarrow{I^{\otimes N-1}\otimes I} \frac{1}{\sqrt{2}}|\widetilde{W}^{N-1}\rangle|0\rangle + \frac{1}{\sqrt{2}}|0\ldots 00\rangle|1\rangle, \\
01:\ |W^N\rangle &\xrightarrow{I^{\otimes N-1}\otimes \sigma_1} \frac{1}{\sqrt{2}}|\widetilde{W}^{N-1}\rangle|1\rangle + \frac{1}{\sqrt{2}}|0\ldots 00\rangle|0\rangle, \\
10:\ |W^N\rangle &\xrightarrow{I^{\otimes N-1}\otimes -i\sigma_2} \frac{1}{\sqrt{2}}|\widetilde{W}^{N-1}\rangle|1\rangle - \frac{1}{\sqrt{2}}|0\ldots 00\rangle|0\rangle, \\
11:\ |W^N\rangle &\xrightarrow{I^{\otimes N-1}\otimes \sigma_3} \frac{1}{\sqrt{2}}|\widetilde{W}^{N-1}\rangle|0\rangle - \frac{1}{\sqrt{2}}|0\ldots 00\rangle|1\rangle.
\end{aligned} \qquad (39)$$

Alice applies the operators selected from $\{I, \sigma_1, -i\sigma_2, \sigma_3\}$ on her qubit. One can readily find that the four states produced are mutually orthogonal. Now Alice can send her qubit to Bob who makes a projective measurement on the N qubits in a basis that includes the four states produced. Since the states produced are orthogonal, Bob can perfectly distinguish which operator Alice has applied. Therefore, Bob can decode two classical bits of information from Alice's encoding.

Note that the two protocols by using state $|W\rangle_{123}$ stated in [12] are the special case of the above generalized protocols when $N = 3$.

C. The optimal way to partition state $|\widetilde{W}^N\rangle$ into two subsystems

As we can see, state $|\widetilde{W}^N\rangle$ has a good property—symmetry that is interesting and useful for some applications. Thus this state may be worthy of further consideration. In the previous subsection, we investigated the case of partitioning it into two subsystems such that one subsystem possesses one qubit and the other subsystem possesses the residual qubits, and then we found that the entanglement between the two subsystems becomes close to 0 when N increases, which may be not a good news for our teleportation and superdense coding. However, if we make light of this state just because of these results, then that may be the really bad news. In the following we will find an optimal partitioning way in the sense that the entanglement between the two subsystems resulted attains

its maximum value.

A general way of partitioning state $|\widetilde{W}^N\rangle$ into two subsystems is to let one subsystem have x ($1 \leq x < N$ and x is an integer number) qubits and the other subsystem the left $N-x$ qubits. Then, due to its symmetry property, no matter what the x qubits are, the state can always be rewritten in the form:

$$\begin{aligned}|\widetilde{W}^N\rangle &= \frac{1}{\sqrt{N}}\Big[\Big(|\overbrace{10\ldots0}^{x}\rangle + \cdots + |0\ldots01\rangle\Big)|\overbrace{0\ldots0}^{N-x}\rangle \\ &\quad + |\overbrace{0\ldots0}^{x}\rangle\Big(|\overbrace{10\ldots0}^{N-x}\rangle + \cdots + |0\ldots01\rangle\Big)\Big] \\ &= \sqrt{\frac{x}{N}}|\widetilde{W}^x\rangle|\overbrace{0\ldots0}^{N-x}\rangle + \sqrt{\frac{N-x}{N}}|\overbrace{0\ldots0}^{x}\rangle|\widetilde{W}^{N-x}\rangle\end{aligned} \tag{40}$$

where states $|\widetilde{W}^x\rangle$ and $|\widetilde{W}^{N-x}\rangle$ are resulted from N in Eq. (29) being x and $N-x$, respectively. Then the entanglement between the two subsystems resulted, say A and B, is the binary entropy given by

$$E_{A|B}(|\widetilde{W}^N\rangle) = -\frac{x}{N}\log_2\frac{x}{N} - (1-\frac{x}{N})\log_2(1-\frac{x}{N}). \tag{41}$$

Thus from the properties of binary entropy, we know that the absolute value $|\frac{x}{N}-\frac{1}{2}|$ is smaller, the value in Eq. (41) is bigger, and the maximum value 1 is attained when $\frac{x}{N} = \frac{1}{2}$. Therefore, there are two cases we should consider:

- (i) N is even. Then the maximum value 1 can always be attained by letting $x = \frac{N}{2}$. That is, in this case, a balanced partition always leads to two subsystems sharing 1 ebit of entanglement.
- (ii) N is odd. In the case, we let $x = \lfloor\frac{N}{2}\rfloor$, that is, x is the integer part of number $\frac{N}{2}$. Then the value $|\frac{x}{N} - \frac{1}{2}|$ is the smallest among all possible x, and thus $E_{A|B}(|\widetilde{W}^N\rangle)$ attains its maximum value among all possible x. Of course, in this case, the entanglement can not attain 1 ebit forever, but when N is large enough, the value can arbitrarily close to 1.

In summary, for state $|\widetilde{W}^N\rangle$, among all the ways of partitioning it into two subsystems, the optimal way is such one that leads to two balanced (or approximately balanced) subsystems. Below we provide an example to apply the way.

1. *An example: state $|\widetilde{W}^4\rangle$ used for teleportation of entangled pairs*

As an example, we consider state $|\widetilde{W}^N\rangle$ for $N = 4$. That is the following state

$$|\widetilde{W}^4\rangle = \frac{1}{\sqrt{4}}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle). \tag{42}$$

At the first blush, it seems impossible to be comparable with the following maximally entangled state

$$|GHZ^4\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle). \tag{43}$$

However, according to what we have obtained before, if we partition it into two subsystems such that every subsystem has two qubits, then we can rewrite it in this form:

$$|\widetilde{W}^4\rangle = \frac{1}{\sqrt{2}}(|\varphi^+\rangle|00\rangle + |00\rangle|\varphi^+\rangle) \tag{44}$$

where $|\varphi^\pm\rangle = \frac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle)$. In this way, the entanglement between the two subsystems is one ebit. At the same time, note that if we partition $|GHZ^4\rangle$ in the same way, then the entanglement between two subsystems resulted is also one ebit. Therefore, in a certain situation, $|\widetilde{W}^4\rangle$ can be used to fulfill the same task as by using $|GHZ^4\rangle$.

Teleportation of entangled pairs via three-particle states was discussed in [8, 9, 10]. Next we see how state $|\widetilde{W}^4\rangle$ is used as a shared resource for teleportation of an entangled pair. Detailedly, Bob and Charlie possess the 3th and 4th qubits, respectively, and Alice possesses the first two qubits. Alice also possesses an entangled pair of qubits given by

$$|\psi\rangle_{ab} = \alpha|00\rangle + \beta|11\rangle, \text{ or } |\psi'\rangle_{ab} = \alpha|01\rangle + \beta|10\rangle. \tag{45}$$

Then we have

$$\begin{aligned}&|\psi\rangle_{ab}|\widetilde{W}^4\rangle \\ &= (\alpha|00\rangle + \beta|11\rangle)\frac{1}{\sqrt{2}}(|\varphi^+\rangle|00\rangle + |00\rangle|\varphi^+\rangle) \\ &= \frac{1}{2}\Big[|\mu^+\rangle(\alpha|00\rangle + \beta|\varphi^+\rangle) + |\mu^-\rangle(\alpha|00\rangle - \beta|\varphi^+\rangle) \\ &\quad + |\omega^+\rangle(\alpha|\varphi^+\rangle + \beta|00\rangle) + |\omega^-\rangle(\alpha|\varphi^+\rangle - \beta|00\rangle)\Big],\end{aligned} \tag{46}$$

and similarly,

$$\begin{aligned}&|\psi'\rangle_{ab}|\widetilde{W}^4\rangle \\ &= (\alpha|01\rangle + \beta|10\rangle)\frac{1}{\sqrt{2}}(|\varphi^+\rangle|00\rangle + |00\rangle|\varphi^+\rangle) \\ &= \frac{1}{2}\Big[|\pi^+\rangle(\alpha|00\rangle + \beta|\varphi^+\rangle) + |\pi^-\rangle(\alpha|00\rangle - \beta|\varphi^+\rangle) \\ &\quad + |\varpi^+\rangle(\alpha|\varphi^+\rangle + \beta|00\rangle) + |\varpi^-\rangle(\alpha|\varphi^+\rangle - \beta|00\rangle)\Big],\end{aligned} \tag{47}$$

where

$$\begin{aligned}|\mu^\pm\rangle &= \frac{1}{\sqrt{2}}(|00\rangle|\varphi^+\rangle \pm |11\rangle|00\rangle), \\ |\omega^\pm\rangle &= \frac{1}{\sqrt{2}}(|00\rangle|00\rangle \pm |11\rangle|\varphi^+\rangle), \\ |\pi^\pm\rangle &= \frac{1}{\sqrt{2}}(|01\rangle|\varphi^+\rangle \pm 10\rangle|00\rangle), \\ |\varpi^\pm\rangle &= \frac{1}{\sqrt{2}}(|01\rangle|00\rangle \pm |10\rangle|\varphi^+\rangle).\end{aligned} \tag{48}$$

The set $S = \{|\mu^{\pm}\rangle, |\omega^{\pm}\rangle, |\pi^{\pm}\rangle, |\varpi^{\pm}\rangle\}$ can be extended to an orthogonal basis of the Hilbert space $H_2^{\otimes 4}$. Now Alice can make a projective measurement in a basis that includes S on the four qubits possessed by her, and sends the results of measurement using three cbits to Bob and Charlie. Then Bob and Charlie can together choose an appropriate joint unitary operator (non-local operator) on their qubits to recover the state $|\psi\rangle_{ab}$ or $|\psi^{'}\rangle_{ab}$ at their qubits. For instance, if the result of measurement is μ^+, then Bob and Charlie make a unitary operation $U = |00\rangle\langle 00| + |\varphi^+\rangle\langle 11| + |11\rangle\langle\varphi^+| + |\varphi^-\rangle\langle\varphi^-|$ on their two qubits.

Note that in the above protocol, the two kinds of states $|\psi\rangle_{ab}$ and $|\psi^{'}\rangle_{ab}$ can be teleported by using the same protocol with three bits of classical communication. Here if we just want to teleport one kind of them, say $|\psi\rangle_{ab}$, then two bits of classical communication is enough. In addition, since the operations done by Bob and Charlie are non-local, Bob and Charlie can not be spatially separated, which may be the limitation of this protocol. On the other hand, because of this non-locality, Bob and Charlie must cooperate to recover state $|\psi^{'}\rangle_{ab}$ or $|\psi\rangle_{ab}$, so this protocol may be considered for *quantum secret sharing* [20]. By the way, state $|\widetilde{W}^4\rangle$ can also be used to transmit three classical bits by sending two qubits.

D. Generalize state $|W\rangle_{123}$ to multi-particle systems with higher dimension

In the following, we consider the state of multi-particle systems with d-dimensional particles, such that it generalizes the state $|W\rangle_{123}$. First we let

$$|\xi_{(i)}^N\rangle = \frac{1}{\sqrt{N}}\big(|\overbrace{i0\ldots0}^{N}\rangle + |0i\ldots0\rangle + \ldots|0\ldots0i\rangle\big) \quad (49)$$

for $i = 1, \ldots, d-1$. Then we let

$$|\Omega^N\rangle = \frac{1}{\sqrt{d}}\Big[\sum_{i=1}^{d-1}|\xi_{(i)}^{N-1}\rangle|i-1\rangle + |0\ldots0\rangle|d-1\rangle\Big]. \quad (50)$$

Now state $|\Omega^N\rangle$ may be taken as a reasonable generalization of state $|W\rangle_{123}$. In fact, Eq. (50) can be regarded as the Schmidt decomposition for the state of bisystem AB, where subsystem A consists of the first $N-1$ particles and subsystem B has the Nth particle. Thus, the entanglement between them is $\log_2 d$. Clearly, when $d = 2$, state $|\Omega^N\rangle$ reduces into state $|W^N\rangle$ given by Eq. (29).

Next we see how state $|\Omega^N\rangle$ is used for superdense coding. Suppose that Alice and Bob share state $|\Omega^N\rangle$ such that Alice possesses subsystem B and Bob possesses subsystem A. Alice can now manipulate her subsystem by unitary operations

$$U(m,n) = \sum_{k=0}^{d-1} e^{2\pi ikm/d}|k\rangle\langle k\oplus n| \quad (51)$$

where $\oplus$ denotes addition modulo d, and $m, n = 0, \ldots, d-1$. After that, Alice sends her subsystem to Bob. Then similar to the case in [18], d^2 mutually orthogonal states will be produced at the disposal of Bob who can make a projective measurement on the N particles to distinguish what operation Alice has applied. Therefore, Bob gets $2\log_2 d$ classical bits of information from Alice.

IV. CONCLUSION

In this work, we investigated that the states of W-class are used for teleportation and superdense coding, and characterized W-class in multi-particle systems. We described two transformations of the shared resources for teleportation and superdense coding, with which we obtained a sufficient and necessary condition for a state of W-class being suitable for perfect teleportation and superdense coding, and we found that the state $|W\rangle_{123}$ can be used to transmit three classical bits by sending two qubits, which was considered to be impossible in [12]. We generalized the states of W-class to multi-qubit systems and multi-particle systems with higher dimension. We proposed two protocols for teleportation and superdense coding by using W-states of multi-qubit systems that generalize the protocols by using $|W\rangle_{123}$ stated in [12]. We obtained an optimal way to partition some W-states of multi-qubit systems into two subsystems, such that the entanglement between them achieves maximum value.

As we pointed out in Sec. II D, although state $|W\rangle_{123}$ can be used to transmit three classical bits by sending two qubits, it may be not suitable for the case of two senders and one receiver. Then in future, one can consider whether there exist W-states suitable for superdense coding with two senders and one receiver. If the answer is yes, then one can further generalize that to the case of multi-sender and one receiver. Otherwise, one should prove the impossibility.

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